The Moonshine Module for Conway's Group
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I'm going to talk about some work I've done with John Duncan over the past couple years. I'll begin with an overview of moonshine and then get into what we've done.

Moonshine refers to a number of amazing connections between the representation theory of finite groups and modular functions; this is a neat thing because these two fields are a priori quite unrelated, so the deep connection in moonshine suggests an underlying structure that is not yet understood.

What is this mysterious connection? There are many examples of moonshine, but I'll focus on one particular example: Conway moonshine. I said that moonshine consists of connections between finite groups and modular functions; the finite group in this case is Conway's group Co_0, the automorphism group of a special 24-dimensional lattice known as the Leech lattice.

Co_0 is a rather large finite group, weighing in at

$$8\,315\,553\,613\,086\,720\,000 \tag{1}$$

(over 8 quintillion) elements; it has 167 irreducible representations, whose dimensions are

$$1, 24, 276, 299, 1771, 2024, 2576, 4576, 8855, \ldots. \tag{2}$$

To understand the modular functions in this example, another bit of background. The upper half plane

$$\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\} \tag{3}$$

equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \tag{4}$$

is a model of the hyperbolic plane, and the group of (orientation-preserving) isometries of this hyperbolic plane is $\mathrm{SL}_2\,\mathbb{R}$ acting by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}. \tag{5}$$

Given a discrete subgroup $\Gamma < \mathrm{SL}_2\,\mathbb{R}$ we can form the orbit space $\Gamma\backslash\mathbb{H}$, a complex surface, and by adding finitely many points we obtain a compact surface $\Gamma\backslash\widehat{\mathbb{H}}$. Meromorphic functions on $\Gamma\backslash\widehat{\mathbb{H}}$ are called *modular functions* for Γ. Equivalently, modular functions for Γ are meromorphic functions $f : \mathbb{H} \to \mathbb{C}$ that satisfy the transformation

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \tag{6}$$

(a function on $\mathbb{H}$ descends to the quotient $\Gamma\backslash\mathbb{H}$ exactly when it is invariant under the action of Γ, cf. (5)). If Γ contains $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, the transformation $\tau \mapsto \tau + 1$, then (6) implies $f(\tau + 1) = f(\tau)$ and therefore f has a Fourier series in terms of $q = e^{2\pi i\tau}$.

$$f(\tau) = \sum_{n \geq N} a_n q^n \qquad\qquad (q = e^{2\pi i\tau}) \tag{7}$$

Define the *genus* of a subgroup $\Gamma < \mathrm{SL}_2\,\mathbb{R}$ to be the genus of $\Gamma\backslash\widehat{\mathbb{H}}$. It is a fact that the field of meromorphic functions on $\Gamma\backslash\widehat{\mathbb{H}}$, i.e. the field of modular functions for Γ, is generated by a single element exactly when the genus of $\Gamma\backslash\widehat{\mathbb{H}}$ is 0 (i.e. Γ has genus 0). In this case, such a generator of the field of modular functions for Γ is called a *principal modulus* for Γ (or Hauptmodul). This generator is not unique, for we can scale it or add constants as we please; if we impose the normalization condition $q^{-1} + 0 + O(q)$ on its Fourier expansion, then it is unique, and we call it a *normalized* principal modulus.

Let's look at an example. The subgroup $\Gamma_0(2) < \mathrm{SL}_2\,\mathbb{R}$ consists of integer matrices of determinant 1 which are upper triangular mod 2.

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}, ad - 2bc = 1 \right\} \tag{8}$$

This is a genus 0 group, and its normalized principal modulus is given by the Fourier expansion

$$f(\tau) = q^{-1} - 0 + 276q - 2048q^2 + 11202q^3 - \cdots \tag{9}$$

(recall $q = e^{2\pi i\tau}$). If you were paying very close attention earlier, these numbers may seem familiar; they are nearly the dimensions of irreducible representations of Co_0. In fact,

$$\begin{aligned} 1 &= 1 \\ 276 &= 276 \\ 2048 &= 2024 + 24 \\ 11202 &= 8855 + 2024 + 299 + 24, \end{aligned} \tag{10}$$

Check this!

where the numbers on the left are coefficients of f and the numbers on the right are dimensions of irreducible representations of Co_0. You've probably seen the famous equation $196884 = 1 + 196883$, and these are the analogs for Conway moonshine.

If we extrapolate from these observations, we may guess that each coefficient of f is the dimension of a representation V_i of Co_0. We can gather these representations in a direct sum to obtain a graded representation

$$V = \bigoplus_{i \geq -1} V_i \tag{11}$$

whose *graded dimension*

$$\dim V = \sum_{i \geq -1} \dim V_i \, q^i \tag{12}$$

is f. The existence of such a representation would explain, in some sense, why the dimensions of representations of Co_0 appear in the Fourier coefficients of f.

We can go further. When one is confronted with a representation it is natural to ask about its character, i.e. the traces of group elements in the representation; or in our case *graded traces*

$$\operatorname{tr}_V g = \sum_{i \geq -1} \operatorname{tr}_{V_i} g \, q^i \tag{13}$$

for each group element g.

Note that the graded trace of the identity is simply the graded dimension (because the trace of the identity is the dimension of the space), and this graded trace is a very special function: the (normalized) principal modulus for the genus 0 group $\Gamma_0(2)$. Using the guesses for decompositions into irreducible representations aboce, we can compute the first few terms of the graded trace of each Co_0 element. If we do this then, amazingly, it appears that the graded characters of all elements of Co_0 are (normalized) principal moduli for genus 0 subgroups of $\mathrm{SL}_2 \mathbb{R}$. This is the main content of Conway moonshine, the relationship between representations of Co_0 and (normalized) principal moduli of genus 0 groups.

But so far this is all speculation. We would like to prove that the representation V exists, and that all graded characters are indeed principal moduli of genus 0 groups. The best way to do this is to construct it explicitly (and find expressions for the graded traces). Even more, we would like a "conceptual understanding", whatever that will turn out to mean, of the connection between our finite group and principal moduli. To have any hope of this, the representation we construct must be more than a vector space; it must have some additional structure.

The way moonshine is developing, "additional structure" seems to mean the structure of a "vertex algebra". A vertex algebra, very roughly, is a vetor space with a multiplication operation that lands not back in the vector space, but in the space of formal Laurent series over the vector space.

$$V \otimes V \to V((z)) \tag{14}$$

An important fact about vertex algebras is that they arose from quantum theory, and they can be used to model certain quantum systems. This connects moonshine with physics, and I'll say more about that later. But I should mention at this point that a "conceptual explanation" of moonshine of the type I referred to above is still missing.

The main result that I wanted to tell you about today is the construction of the vertex algebra, which we call $V^{s\natural}$, that furnishes the representation linking Conway's group to the principal moduli associated by moonshine. I won't go into much detail in the actual construction; the important thing I want to convey is that it's simple. The reason this is important is the hope that Conway moonshine can serve as a sort of stepping-stone to monstrous moonshine, where the corresponding construction is quite complicated.

There is a standard method of producing a vertex algebra from a (finite-dimensional complex) vector space (equipped with a (non-degenerate symmetric) bilinear form), via Clifford algebras. Apply this construction to a 24-dimensional space $\mathfrak{a}$ to obtain a vertex algebra $A(\mathfrak{a})$. In fact we obtain a super vertex algebra, meaning that it decomposes into even and odd subspaces.

$$A(\mathfrak{a}) = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1 \tag{15}$$

This vertex has a canonical vertex algebra module (obtained by a similar Clifford algebra construction) with a similar decomposition

$$A(\mathfrak{a})_{\mathrm{tw}} = A(\mathfrak{a})^0_{\mathrm{tw}} \oplus A(\mathfrak{a})^1_{\mathrm{tw}}. \tag{16}$$

Using the module structure and certain facts about vertex algebra, we can give a vertex algebra structure to $A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1_{\mathrm{tw}}$; this completes our construction.

$$V^{s\natural} = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1_{\mathrm{tw}} \tag{17}$$

The construction of this representation realizing the moonshine phenomenon between Conway's group and principal moduli of genus 0 groups is an important step toward a conceptual understanding of moonshine.

Very quickly I want to mention a neat application to physics of this construction. The vertex algebra $V^{s\natural}$ has a canonical vertex algebra module $V^{s\natural}_{\mathrm{tw}}(= A(\mathfrak{a})^1 \oplus A(\mathfrak{a})^0_{\mathrm{tw}})$ which also receives an action from Conway's group. Very recently Gaberdiel–Hohenegger–Volpato showed that a particular class of quantum field theories, viz. K3 sigma models, all have automorphism group a subgroup of Co_0. Now such a quantum field theory has an important piece of data called its elliptic genus, a sort of partition function, and this elliptic genus can be "twined" by a symmetry of the field theory to produce a twined elliptic genus that encodes important data about the field theory. The amazing thing is that the graded characters of Co_0 elements acting on $V^{s\natural}_{\mathrm{tw}}$ appear to be exactly the twined elliptic genera of this class of field theories, K3 sigma models. But all that will need it's own talk.

Thanks for listening!